



Revisiting parton evolution and the large- x limit

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Received 18 January 2006; accepted 9 February 2006

Available online 17 February 2006

Editor: N. Glover

Abstract

This remark is part of an ongoing project to simplify the structure of the multi-loop anomalous dimensions for parton distributions and fragmentation functions. It answers the call for a “structural explanation” of a “very suggestive” relation found by Moch, Vermaseren and Vogt in the context of the $x \rightarrow 1$ behaviour of three-loop DIS anomalous dimensions. It also highlights further structure that remains to be fully explained.

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1. Introduction

This Letter stems from a project to better understand the structure of multi-loop anomalous dimensions both for parton distributions and fragmentation functions [1]. These distributions, which we shall generically denote as D ($D_N(Q^2)$ in moment space, $D(x, Q^2)$ in x -space) satisfy a renormalisation group equation

$$\frac{dD_N(Q^2)}{d \ln Q^2} \equiv \partial_t D = \gamma(N, \alpha_s(Q^2)) D_N(Q^2), \quad (1)$$

where ∂_t is a compact notation for the derivative with respect to $t = \ln Q^2$ and $\gamma(N, \alpha_s)$ are elements of an anomalous dimension matrix. The latter have been calculated in terms of an expansion in the coupling α_s up to three (two) loops in the space-like (time-like) cases [2–5]. They become increasingly cumbersome beyond leading order.

Conventionally one defines parton splitting functions, $P(x)$, as the inverse Mellin transform of the corresponding anomalous dimensions, giving evolution equation in x space in terms of a

direct convolution

$$\partial_t D(x, Q^2) = \int_0^1 \frac{dz}{z} P(z, \alpha_s(Q^2)) D\left(\frac{x}{z}, Q^2\right), \quad (2)$$

where $D(x, Q^2)$ has the physical support $x \leq 1$. We have reason to suspect that there might exist a reformulation of the evolution equations (2) in which, by generalising the structure on the right-hand side, one is able to simplify the splitting functions. This is equivalent to stating that the higher-loop structure of the anomalous dimensions in (1) can in part be understood as inherited from non-linear combinations of lower loops.

In this Letter, the new splitting functions would not only be more compact, but they would also exhibit some important physical properties: beyond first loop they should vanish at large x , and they should be identical for space-like and time-like evolution, thus restoring Gribov–Lipatov reciprocity [6] (broken beyond first loop in the standard formulation, see [4]).

A possible reformulation of (2) is²

$$\partial_t D(x, Q^2) = \int_0^1 \frac{dz}{z} \mathcal{P}(z, \alpha_s(z^{-1} Q^2)) D\left(\frac{x}{z}, z^\sigma Q^2\right), \quad (3)$$

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² Such a reformulation of the notion of parton splitting functions originally proposed in [7] has been carried out in detail in the context of heavy-quark fragmentation functions, where it was found to greatly improve the perturbative series [8].

where $\sigma = -1$ (+1) for the space-like (time-like) case. The choice of $z^\sigma Q^2$ as the logarithmic ordering parameter (parton evolution ‘time’) corresponds to ordering parton splittings in the *fluctuation lifetimes* of successive virtual parton states. Assuming the new $\mathcal{P}(x)$ splitting functions to be identical in the space-like and time-like cases one obtains that the ‘traditional’ two-loop splitting functions in the two cases, $P^{(2),S}(x)$ and $P^{(2),T}(x)$ differ by

$$\frac{1}{2}(P_{ns}^{(2),T} - P_{ns}^{(2),S}) = \int_0^1 dz \int_0^1 dy \delta(x - yz) P_{qq}^{(1)}(z) \ln z \{P_{qq}^{(1)}(y)\}_+. \quad (4)$$

This is precisely the relation that was noted by Curci, Furman-ski and Petronzio in [4] for non-singlet quark evolution. In the singlet case (both for quarks and gluons) there are also interesting patterns, see Appendix A. Furthermore, if one writes $\mathcal{P}(z, \alpha_s)$ as a series in the physical coupling, $\alpha_{\text{ph}} = \alpha_{\text{MS}} + \frac{K}{4\pi} \alpha_{\text{MS}}^2 + \dots$ (with $K = (\frac{67}{9} - \frac{\pi^2}{3})C_A - \frac{10}{9}n_f$), the two-loop part of $\mathcal{P}(x, \alpha_s)$ vanishes as $(1-x)$ for $x \rightarrow 1$. This corresponds to the wisdom of Low, Burnett and Kroll [9] according to which the classical nature of soft radiation reveals itself at the level of the $1/(1-x)$ and constant terms. This classical nature of soft radiation allows one to absorb all soft singularities into the first loop and to look upon higher-loop splitting functions as due to true multi-parton (quantum) fluctuations.

Finally, one notes also the z -dependence of the argument of the coupling in (3). Its trace is visible in the explicit structure of all the diagonal two-loop anomalous dimensions. Moreover, this argument naturally emerges when using dispersive reasoning to carry out a careful treatment of the appearance of the running coupling in inclusive processes [10].

We are still far from a good understanding of how to simplify the structure of multi-loop anomalous dimensions, notably because of complications that arise from off-diagonal transitions (see Appendix A). Nevertheless the belief that the $x \rightarrow 1$ limit is under control has led us to investigate the implications of Eq. (3) for the large- N structure of three loop anomalous dimensions.

As we shall see this will provide insight concerning a “very suggestive” relation noted by Moch, Vermaseren and Vogt (MVV) which, in their words, “seems to call for a structural explanation”: in both the non-singlet quark [2] and diagonal singlet quark–quark and gluon–gluon three-loop splitting functions [3], they observed that the third-loop coefficients, C_3^a , in the large- N expansion of the n -loop anomalous dimensions,³

$$\gamma_{aa}(N) = -A^a (\ln N + \gamma_e) + B^a - C^a N^{-1} \ln N + \mathcal{O}(N^{-1}),$$

$$\gamma_{aa} \equiv \sum \gamma_{n,aa} \left(\frac{\alpha_{\text{MS}}}{4\pi} \right)^n, \quad A^a \equiv \sum A_n^a \left(\frac{\alpha_{\text{MS}}}{4\pi} \right)^n, \quad \text{etc.} \quad (5)$$

³ We define $\gamma_n(N) = \int_0^1 dz z^{N-1} P^{(n)}(z)$; this has the opposite sign to the convention of MVV; we have also changed the sign in front C^a as compared to Eq. (3.10) of Ref. [2], which contains a misprint.

are simply related with A_2^a and A_1^a , i.e., $C_3^a = 2A_1^a A_2^a$, where $a = q, g$. This supplements the two-loop relation, $C_2^a = (A_1^a)^2$ [4].

2. MVV relation

For the purpose of studying the $x \rightarrow 1$ limit we initially approximate $\mathcal{P}(x, \alpha_s)$ by the product of the physical coupling, α_{ph} , and the 1-loop splitting function, $\mathcal{P}(x, \alpha_s) \simeq (\alpha_{\text{ph}}/4\pi) P^{(1)}(x)$ (for compactness we will write $(\alpha_{\text{ph}}/4\pi) \equiv \alpha$). To deal with the correlated z and Q^2 dependences in the right-hand side of (3), we rewrite it as

$$\partial_t D(x, Q^2) = \int_0^1 \frac{dz}{z} P^{(1)}(z) [e^{\ln z \beta(\alpha) \partial_\alpha} \alpha] \left[e^{\sigma \ln z \partial_t} D\left(\frac{x}{z}, Q^2\right) \right], \quad (6)$$

where both α and D are now evaluated at scale Q^2 and $\beta(\alpha) \equiv -d\alpha/dt = -\sum_{n=0} \beta_n \alpha^{n+2}$. The Mellin transform of this equation results in the formal expression,

$$\partial_t D_N = \gamma_1(N + \beta(\alpha) \partial_\alpha + \sigma \partial_t) \alpha D_N, \quad (7)$$

where $\gamma_1(N)$ is the Mellin transform of the first order splitting function $P^{(1)}(x)$. We note that ∂_t operates only on D_N and not on α . This gives an all-order model for the anomalous dimension, $\gamma(N) \equiv D_N^{-1} \partial_t D_N$.

Expanding (7) results in

$$\begin{aligned} \gamma &\equiv \gamma[\alpha] \\ &= \alpha \gamma_1 + \dot{\gamma}_1 D^{-1} (\beta \partial_\alpha + \sigma \partial_t) (\alpha D) \\ &\quad + \frac{1}{2} \ddot{\gamma}_1 D^{-1} (\beta \partial_\alpha + \sigma \partial_t)^2 (\alpha D) + \dots \\ &= \alpha \gamma_1 + \dot{\gamma}_1 (\beta + \sigma \alpha \gamma) \\ &\quad + \frac{1}{2} \ddot{\gamma}_1 D^{-1} (\beta \partial_\alpha + \sigma \partial_t) (\beta D + \sigma \alpha D \gamma) + \dots \\ &= \alpha \gamma_1 + \dot{\gamma}_1 (\beta + \sigma \alpha \gamma) \\ &\quad + \frac{1}{2} \ddot{\gamma}_1 [\alpha \gamma^2 + \sigma (2\beta \gamma + \alpha \beta \partial_\alpha \gamma) + \beta \partial_\alpha \beta] + \mathcal{O}(\alpha^4), \end{aligned} \quad (8)$$

where dots indicate derivatives with respect to N and $\gamma_1 \equiv \gamma_1(N)$. Solving this iteratively produces

$$\begin{aligned} \gamma &= \alpha \gamma_1 + \alpha^2 \dot{\gamma}_1 (\beta_0 + \sigma \gamma_1) \\ &\quad + \alpha^3 \left[\dot{\gamma}_1 (\beta_1 + \sigma \dot{\gamma}_1 (\beta_0 + \sigma \gamma_1)) \right. \\ &\quad \left. + \frac{1}{2} \ddot{\gamma}_1 (\gamma_1^2 + 3\sigma \beta_0 \gamma_1 + 2\beta_0^2) \right] + \mathcal{O}(\alpha^4). \end{aligned} \quad (9)$$

For the purpose of understanding the MVV relation it suffices to take $\gamma_1 = -A_1 \ln N + \mathcal{O}(1)$ and to keep in (9) only the term $\propto \dot{\gamma}_1 \gamma_1$, giving

$$\gamma = -\alpha A_1 \ln N + \text{const} + \sigma \alpha^2 A_1^2 \frac{\ln N}{N} + \mathcal{O}(N^{-1}). \quad (10)$$

Recalling that $A_1^q \alpha \equiv A^a$ (with $A_1^q = 4C_F$ and $A_1^g = 4C_A$), we can then write the following all-order relation between C^a

and A^a ,

$$C^a = -\sigma (A^a)^2, \quad (11a)$$

or equivalently, in terms of the expansion coefficients C_n ,

$$\begin{aligned} C_1 &= 0, & C_2 &= -\sigma A_1^2, & C_3 &= -2\sigma A_1 A_2, \\ C_4 &= -\sigma (A_2^2 + 2A_1 A_3), & \text{etc.} \end{aligned} \quad (11b)$$

where we have suppressed the index $a = q, g$. For the space-like case ($\sigma = -1$) this explains the MVV observation. For the time-like case we have only the two-loop result [4] to compare to, and it agrees.

3. Pushing our luck

Motivated by the idea that the universality of soft gluon emission holds both in singular and constant terms in gluon energy [9], one may attempt to trace further terms of the large- N expansion generated by Eq. (3). We definitely expect this push to fail at the level of $1/N^2$ (possibly modulo logarithms, see below) because this corresponds to ‘quantum’ terms in the splitting function, which vanish as $1-x$. However we would expect to have control over the $1/N$ term in the anomalous dimension,

$$\begin{aligned} \gamma(N) &= -A(\psi(N+1) + \gamma_e) + B - C(\psi(N) + \gamma_e)N^{-1} \\ &\quad + DN^{-1} + \mathcal{O}(N^{-2} \log^p N). \end{aligned} \quad (12)$$

Compared to (5) we have shifted the argument of the logarithm in the A -term, $N \rightarrow N+1$, added the constant γ_e in the C -term and then replaced logarithms with ψ functions. These modifications do not affect the first three functions A , B and C but serve to simplify the next subleading term $\propto 1/N$. Additionally they lead to a compact x -space image of (12),

$$\begin{aligned} P(x) &= \frac{Ax}{(1-x)_+} + B\delta(1-x) + C \ln(1-x) + D \\ &\quad + \mathcal{O}((1-x) \log^p(1-x)). \end{aligned} \quad (13)$$

Here, the presence of $x/(1-x)$ in the first term (as opposed to $1/(1-x)$) is a consequence of Low’s theorem [9]. In the calculation of the C coefficients we could safely ignore the $\mathcal{O}(1)$ piece of γ_1 . This is no longer possible when calculating D because, in our non-linear construction, this constant (*unity* in Mellin space) is multiplied by $\dot{\gamma}_1 \sim 1/N$ thus contributing to D . The extension of (9) to account for B in all orders is obtained by generalising

$$\begin{aligned} \alpha\gamma_1 &\rightarrow \alpha(\gamma_1 - B_1) + B \rightarrow -A(\psi(N+1) + \gamma_e) \\ &\quad + B + \mathcal{O}(N^{-2}). \end{aligned} \quad (14)$$

The reason why the structure of the Taylor expansion (9) is unmodified modulo this simple substitution is that B , being a constant, disappears everywhere but undotted factors of γ_1 .

This leads to the following all-order expectation for D ,

$$D^a = A^a \left(\frac{\partial_t A^a}{A^a} - \sigma B^a \right), \quad (15)$$

where we have rewritten the $A_1^a \beta$ term that comes from Eq. (8) as $-(\partial_t A^a)$ (recalling the definition of β in terms of the physical coupling). The $\overline{\text{MS}}$ expansion for D^a is then

$$\begin{aligned} D_1 &= 0, & D_2 &= -A_1(\sigma B_1 + \beta_0), \\ D_3 &= -A_1(\sigma B_2 + \beta_1) - A_2(\sigma B_1 + 2 \cdot \beta_0), & \text{etc.} \end{aligned} \quad (16)$$

Hard luck. Examining the full known results for the two and three-loop splitting functions, we find agreement for D_2 (space and time-like, and quark and gluon channels); however the result for the space-like D_3 is as follows (for both quarks and gluons)

$$D_3 = A_1(B_2 - \beta_1) + A_2(B_1 - 1 \cdot \beta_0). \quad (17)$$

There is one mismatch between Eqs. (16) and (17), which we have highlighted in boldface. Had (15) contained $(\partial_t \alpha_{\text{MS}})/\alpha_{\text{MS}}$ instead of $(\partial_t A^a)/A^a$ we would have obtained agreement with the full result for D_3 , however we see no reason why it should be the $\overline{\text{MS}}$ coupling that appears there instead of the physical coupling (which is equivalent to putting A^a). The remarkable simplicity of the mismatch calls for a further *structural explanation*.

Good luck. Despite the disagreement in the comparison with the exactly calculated subleading D term (which we hope can be understood) we have also investigated the coefficient of terms that vanish for $x \rightarrow 1$ but that are logarithmically enhanced there. We have found agreement using (9) for the $\alpha_s^2(1-x) \ln(1-x)$ and the $\alpha_s^3(1-x) \ln^2(1-x)$ terms in the (space-like) non-singlet anomalous dimensions, while $\alpha_s^3(1-x) \ln(1-x)$ contains structures that remain to be understood. As for the diagonal singlet anomalous dimensions, at three loops the coefficient of $\alpha_s^3(1-x) \ln^2(1-x)$ agrees only in its n_f -independent parts and there is an additional unexpected $\alpha_s^3(1-x) \ln^3(1-x)$ contribution proportional to n_f , whose origin may only be explained once the higher-order structure of the off-diagonal splittings is elucidated.

The wealth of structure that is present in higher-order splitting functions is suggestive of underlying simplicity. Possible sources of such simplicity, as proposed here, are the universal nature of soft gluon radiation and the reformulation of the notion of parton splitting functions with the aim of preserving universality between space and time-like parton multiplication (Gribov–Lipatov reciprocity). Whether this picture can be made fully consistent remains to be seen. We look forward to future work shedding more light on this question.

Acknowledgements

We wish to thank Andreas Vogt for numerous discussions on the subject of splitting functions as well as comments on the manuscript. The comparisons performed here would not have been possible without the explicit code provided in the arXiv versions of [2,3] and Fortran [11] and Mathematica [12] packages for dealing with harmonic polylogarithms. We additionally thank Einan Gardi and Alberto Guffanti for a helpful comment, and Werner Vogelsang for bringing to our attention Ref. [14]. Its detailed study of Gribov–Lipatov reciprocity in the second

loop should help elucidate the structure of non-diagonal transitions in Eq. (3).

Appendix A

We do not know how to generalise (3) to non-diagonal transitions. These are considerably more divergent at large x [13] than would be expected based on the non-linear relations that we propose here, going as $\alpha_s^n \ln^{2n-2}(1-x)$ at large x . We suspect that the origin of these additional logarithms may be that in the $\overline{\text{MS}}$ factorisation scheme, for $a \rightarrow b$ transitions with $a \neq b$, the splitting functions could pick up residues from ratios of non-cancelling divergent Sudakov exponents (as well as from singular integrals of these ratios). This belief is not inconsistent with the MVV observation that in the supersymmetric case most of these logarithmic enhancements cancel, since in this case the Sudakov exponents become identical for quarks and gluons and do cancel.

These problems of non-diagonal terms may be responsible for the following fact: at two loops, the \mathcal{P} that appears in (3) for *gluon–gluon* splitting is universal (identical for space and time-like cases) only for two of the colour structures, C_A^2 and C_{An_f} . The analogue of (4) for the remaining colour structure, C_{Fn_f} relates gluon–gluon and singlet quark–quark

splittings. On the left-hand side one finds the combinations $P_{gg}^{(2),T} - P_{qq}^{(2),S}$ and $P_{qq}^{(2),T} - P_{gg}^{(2),S}$, while on the right-hand side one has convolutions involving $P_{gq}^{(1)}(x/z) \cdot \ln z P_{qg}^{(1)}(z)$ and $P_{qg}^{(1)}(x/z) \cdot \ln z P_{gq}^{(1)}(z)$ [7].

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